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Method of Weighted Residuals Applied to Free Shear Layers

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The purpose of this Note is to present a detailed application of the method of weighted residuals (MWR) to laminar incompressible free shear layers. In the past, the MWR has been applied to laminar boundary-layer flows with and without suction and blowing,[1–3] and to turbulent boundary layers.[4] These previous works have all been oriented to attached boundary layers and not to free shear layers or jets. It was decided to investigate the laminar free shear layer as an initial step in a study of turbulent jets by the MWR since an exact solution is available[5] and since the physics of the problem is simple, one is allowed to easily examine the disadvantages of the MWR.

One of the particular questions to be investigated concerns the choice of the velocity profile polynomial and the weighting function, both of which are used in the integral method. A certain arbitrariness is present in the MWR; specifically, Bethel[1,3] uses several different forms of the velocity profile depending on the external flowfield. Unfortunately, there is no way to determine beforehand what is the proper choice of approximating function.

The equations to be solved are the same as those given by Bethel[1] with the exception that the pressure gradient parameter $\dot{U}/U = 0$. Thus

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty} h_i(u)u\,d\eta - \int_{-\infty}^{+\infty} h_i'(u)\frac{\partial^2 u}{\partial \eta^2}\,d\eta = 0 \qquad i = 1,2,\ldots N$$

where $\xi \equiv x/L$, $\eta \equiv Re_\infty^{1/2}y/L$, $u \equiv \bar{u}/u_\infty$. At this point, for the free shear layer problem, we restrict $h_i(u)$ to be lim $\eta \to -\infty$, $h_i(u) = 0$. We then make the usual change of variable $\theta(\xi,u) = 1/(\partial u/\partial\eta)$ and integrate the second term by parts to obtain

$$\frac{d}{d\xi}\int_0^1 h_i(u)u\theta du + \int_0^1 h_i''(u)\frac{1}{\theta}\,du = 0 \qquad i = 1,2,\ldots N$$

where θ is some appropriate form of inverse shear stress profile and $h_i(u)$ is an as yet unspecified weighting function.

The boundary conditions on the shear stress dictate that $\theta \to \infty$ as $u \to 0$ and as $u \to 1$; i.e., there are poles at $u = 0$ and 1. The simplest form of an assumed shear-stress distribution is $\theta(\xi,u) \propto 1/[u(1-u)]$. Admittedly, there are many other forms of θ which will satisfy the required conditions and this is the first point where a decision must be made. We adhere to the belief that one should always try the simplest case that meets all required conditions. Of course, if this approach fails, perhaps one should re-examine the required conditions—they may be incomplete. Since the problem admits to a similar solution and there is an exact solution of this form available,[5] we follow Bethel[1] in saying

$$\theta(\xi,u) = \frac{\xi^{1/2}}{u(1-u)}\sum_{j=1}^{N} c_j u^{j-1}$$

where the c_j are the unknown constants.

The second decision to be made in the solution is what should be the form of the weighting function. The simplest general choice appears to be

$$h_i(u) = u^{mi+n}(1-u)^{ki+l}$$

where m,n,k, and l are constant integers. Bethel[1] points out in the boundary-layer problem that if $h_i(u) = (1-u)^i$ is used and that if $i = 1$, then the resulting integral equation is the Karman-Pohlhausen equation. This occurrence at least assures one that he is using the conservation relations in the solution of the problem. No such occurrence is possible in the free shear layer problem because the integral momentum equation has to be employed as an auxiliary equation to determine the position of the dividing stream line; it can not be used in the system of equations employed to find the unknown c_j. Thus the system of equations contains arbitrary moments of the velocity profile unrelated to physical quantities. For this reason, it seems unwise to try any more complicated weighting function than that suggested before, at least for the present. The one task that the suggested form of $h_i(u)$ can accomplish is to simplify the integrals that have to be carried out. So, with ease of integration as a goal, the final form of the algebraic equations becomes

$$\sum_{1}^{N} c_j\Gamma(mi+j+n)\Big/\Gamma(mi+ki+j+n+l) + 2(ki+l)\sum_{1}^{N} b_j\Gamma(mi+n+j-1)\Big/\Gamma(mi+ki+j+n+l+1)\times[-(2m+k)ij+kij^2+(1-l-2n)j+(l-1)j^2] = 0 \qquad i = 1,2,\ldots N$$

where Γ indicates the gamma function and where the b_j term arises from the same substitution employed by Bethel.[1] The collocation scheme for relating the b_j to the c_j is

$$u_p = (p-1)/(N-C),\ p = 1,2,\ldots N,\ 0 \leqslant C \leqslant 1$$

The constant C is arbitrary except as specified and serves to alter the collocation points. This parameter has been inserted in order to study the effect of the collocation scheme on the convergence of the solution.

The solution of the free shear layer problem would not be complete without obtaining the velocity profiles of u and v in terms of the physical plane coordinates. The procedure

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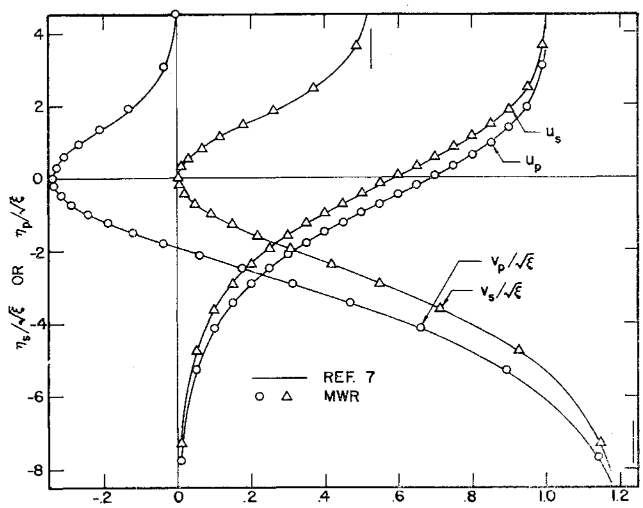

Fig. 1 Velocity profiles for similarity and physical planes.

followed is that used by Casarella and Chou.[6] The dividing stream line displacement distance, $S = Re_\infty^{1/2}\delta^*/L$ is given by a mass balance and the dividing stream line velocity u_* is given by a momentum balance.

This information is sufficient to solve the free shear layer problem in the physical plane. We have a system of N equations in N unknowns which are solved by a Newton-Raphson iteration technique where the incremental values of c_j from step to step in the iteration are obtained by a matrix inversion (a standard Gauss-Jordan routine).

The equations were solved exactly for $N = 2$ (not by an iteration technique) so that a check on the numerical accuracy could be obtained and because early iterative solutions revealed that, for certain weighting functions, solutions for the c_j did not seem to exist. The results of this study are given in Table 1. Only a representative group of choices of $h_i(u)$ are shown. Cases for which $m,n,k,l = 2$ are little different than those shown since the exponent 2 amounts to a neglect of every other moment equation.

Since the solution procedure is iterative, an initial guess for the c_j must be made and this guess must be reasonably close to the actual values. A study was made of the convergence properties of the solution for various initial guesses and it was found that the procedure occasionally converged to the wrong answers in terms of physically possible answers. The conclusion of this study was that some selection process has to be built into the procedure; it was found that the correct answer always resulted in $\Delta_N \simeq 1/\theta_N$ as required, and that incorrect c_j values resulted in Δ_N values nowhere near $1/\theta_N$. This fact was used to decide whether the iteration for the c_j was converging properly.

Furthermore, the physics of the flow dictates that $c_1 > 0$ and $\Sigma c_j > 0$. This restriction occurs because the shear stress

Table 1 Exact solution of equation for $N = 2$

$h_i(u)$				Collocation scheme 1 $C = 0$		Collocation scheme 2 $C = 1$	
m	n	k	l	c_1	c_2	c_1	c_2
0	1	1	0	1.8974	−1.2649	1.8680	−1.0477
1	0	1	0	[a]	[a]	[a]	[a]
1	0	0	1	1.8974	−1.2649	1.8680	−1.0477
0	1	1	1	1.7254	−0.8360	1.6955	−0.7001
1	1	0	1	2.2471	−1.7801	2.4060	−1.6625
1	0	1	1	[a]	[a]	[a]	[a]
1	1	1	0	2.7799	−2.5682	4.3817	−3.8339
1	1	1	1	[a]	[a]	[a]	[a]

[a] No physically meaningful or useful solution.

Table 2 Convergence of displacement thickness S and dividing streamline velocity, $u_*(C = 1)$

$hi(u)$									
m	n	k	l	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
1	0	0	1						
	$S/\xi^{1/2}$			0.5296	0.5598	0.5314	0.5287	0.5284	0.5285
	u^*			0.6321	0.5882	0.5931	0.5904	0.5889	0.5880
1	1	0	1						
	$S/\xi^{1/2}$			0.4587	0.6379	0.5228	0.5368	0.5223	0.5331
	u^*			0.6321	0.5699	0.5990	0.5921	0.5913	0.5862

at $\eta = +\infty$ and $-\infty$ must approach zero positively. There are no further constraints on the c_j so the choice of the initial values is not particularly easy. Some experimenting with the numerical solution is necessary and there does not seem to be any way to avoid this every time a new problem is considered.

The convergence of S and u_* is illustrated in Table 2 for two different weighting functions. The values for u_* in the similarity plane at $N = 4$ are within 2 percent of the exact value obtained by Chapman[5] ($u_* = 0.587$). In Fig. 1 the velocity profile for $h_i(u) = u^i(1 - u)$, $N = 5$ is shown along with the solution obtained by Casarella and Chou.[6] These results compare well with those of Ref. 7 which uses the same method as the present one, the only difference being in the selection of the collocation points.

Several conclusions can be drawn from this preliminary study. 1) The MWR can be accurate to a few percent for N = 4–5 but beyond $N = 5$ convergence is difficult to observe due to the error introduced in the matrix inversion process. An improved matrix inversion procedure is necessary and is currently being studied. 2) It would be preferable for the conservation equations (specifically, momentum) to be included as one of the moment equations, but lack of this does not seem to impair the convergence of the solution. 3) The selection of the initial guesses for the c_j for the iteration procedure can be important and further studies must be made. The MWR suffers from a lack of knowledge of initial guesses for the unknown coefficients. 4) The solution of the free shear layer problem for turbulent flow should be relatively easy to obtain and so a more significant test of the MWR would be to solve the free shear layer problem with an arbitrary initial velocity profile. 5) The MWR is particularly easy to program for a computer and the solution of N equations is only a matter of telling the computer what value N should take.

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